EC 3210 Solutions

Assignment 9

9.1. A resonator mode is found to burn holes in a gas laser gain curve at the FWHM points. If the laser operates at 820 nm and the linewidth is 2 GHz, find the velocity of the atoms or molecules involved in the hole–burning.

We are given a Doppler-broadened line centered at 820 nm with a FWHM of 2 GHz as shown in Fig. 1. We want to find the velocity of the atoms that will be hole-burned at $\nu' = \nu_0 + (\Delta\nu/2)$ and $\nu' = \nu_0 - (\Delta\nu/2)$.

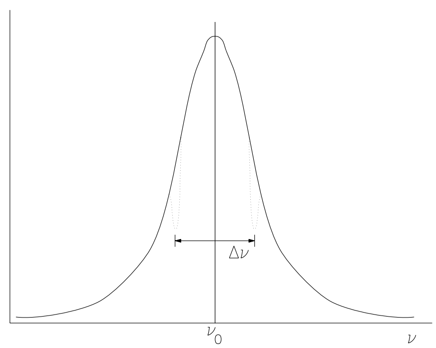

Figure 1: Hole-burning in Prob. 9.1.

We have

$$\nu' - \nu_0 = \frac{\Delta\nu}{2} = \frac{2 \times 10^9}{2} = 1 \text{ GHz}. \tag{1}$$

So we find that

$$v'_z = \mp \frac{c(\nu' - \nu_0)}{\nu_0} = \mp\lambda_0(\nu' - \nu_0) = \mp(820 \times 10^{-9})(1 \times 10^9) = \mp 820 \text{ m} \cdot \text{s}^{-1}. \tag{2}$$

10.1. Suppose a Gaussian plane wave has a spot size of 5 mm at a wavelength of 632.8 nm. Find z_R, $w(z)$, and $R(z)$ at ...

a. ... $z = 10$ mm.

b. ... $z = 10$ cm.

c. ... $z = 10$ m.

d. ... $z = 1$ km.

e. . . . $z = 10$ km.

We are given a Gaussian *plane* wave with $w_0 = 5 \times 10^{-3}$ at $\lambda = 632.8 \times 10^{-9}$.
The Rayleigh range is the same for all propagation distances and is given by

$$z_R = \frac{\pi w_0^2}{\lambda} = \frac{\pi (5 \times 10^{-3})^2}{(632.8 \times 10^{-9})} = 124.1 \text{ m}\,. \tag{3}$$

a. For $z = 10$ mm, we are in the near–field and

$$w(z) \approx w_0 = 5 \text{ mm}\,, \tag{4a}$$

and

$$R(z) \approx \infty\,. \tag{4b}$$

b. For $z = 10$ cm, we are still in the near–field and

$$w(z) \approx w_0 = 5 \text{ mm}\,, \tag{5a}$$

and

$$R(z) \approx \infty\,. \tag{5b}$$

c. For $z = 10$ m, we are in the transition region and

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{Z_R}\right)^2} = w_0 \sqrt{1 + \left(\frac{10}{124.1}\right)^2} = 5.02 \text{ mm} \tag{6a}$$

and

$$R(z) = z + \frac{z_R^2}{z} = 10 + \frac{(124.1)^2}{10} = 1.55 \times 10^3 \text{ m}\,. \tag{6b}$$

d. For $z = 1$ km, we are still in the transition region and

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{Z_R}\right)^2} = w_0 \sqrt{1 + \left(\frac{1 \times 10^3}{124.1}\right)^2} = 40.6 \text{ mm} \tag{7a}$$

and

$$R(z) = z + \frac{z_R^2}{z} = 1 \times 10^3 + \frac{(124.1)^2}{1 \times 10^3} = 1.015 \times 10^3 \text{ m}\,. \tag{7b}$$

e. For $z = 10$ km, we are in the far-field and

$$w(z) \approx \frac{\lambda z}{\pi w_0} = \frac{w_0 z}{z_R} = \frac{(5 \times 10^{-3})(1 \times 10^4)}{124.1} = 40.3 \text{ mm}\,, \tag{8a}$$

and

$$R(z) \approx z = 1.000 \times 10^3 \text{ m}\,. \tag{8b}$$

10.3. Consider a beam collimator with an output aperture of 1 m that produces a Gaussian plane wave at its output plane. The wavelength is 300 nm.

a. Calculate the distance at which the far-field begins.

b. Calculate the beam divergence.

c. Repeat parts a and b if $\lambda = 10.6\ \mu$m.

We are given a beam collimator with an output aperture of 1 m diameter. A Gaussian plane wave is emitted and $\lambda = 300 \times 10^{-9}$.

a. The far-field begins at $10z_R$, so we first need to find z_R. We know that the aperture should be at least 3 spot sizes, so we will assume that the aperture diameter D equals three spot sizes (w_0). The spot size is

$$w_0 = \frac{D}{3} = \frac{1}{3} = 0.33 \text{ m}\,. \tag{9}$$

The Rayleigh range is, then,

$$z_R = \frac{\pi w_0^2}{\lambda} = \frac{\pi(0.333)^2}{300 \times 10^{-9}} = 1.161 \times 10^6 \text{ m} = 1,161 \text{ km}\,. \tag{10}$$

The far-field begins, then, at approximately 11,160 km from the source.

b. The beam divergence in the far-field is

$$\Phi = 2\tan^{-1}\left(\frac{\lambda}{\pi w_0}\right) = 2\tan^{-1}\left(\frac{300 \times 10^{-9}}{\pi(0.333)}\right) = 0.578 \quad \mu\text{r}\,. \tag{11}$$

c. ... if $\lambda = 10.6 \times 10^{-6}$? We know that

$$\frac{z_R(\lambda_1)}{z_R(\lambda_2)} = \frac{\lambda_2}{\lambda_1} \tag{12a}$$

$$\begin{aligned} z_R[10.6 \times 10^{-6}] &= \left(\frac{300 \times 10^{-9}}{10.6 \times 10^{-6}}\right)(z_R[300 \times 10^{-9}]) \qquad (12b)\\ &= \left(\frac{300 \times 10^{-9}}{10.6 \times 10^{-6}}\right)(1.161 \times 10^5) \\ &= 328,000 \text{ m} = 328 \text{ km}\,. \end{aligned}$$

At 10.6 μm, the far-field will start at approximately 3,280 km from the source. In the far-field the beam divergence will be

$$\frac{\Phi(\lambda_1)}{\Phi(\lambda_2)} = \frac{\lambda_1}{\lambda_2} \tag{13a}$$

$$\begin{aligned} \Phi[10.6 \times 10^{-6}] &= \left(\frac{10.6 \times 10^{-6}}{300 \times 10^{-9}}\right)(\Phi[300 \times 10^{-9}]) \qquad (13b)\\ &= (35.3)(0.578)\ \mu\text{r} = 20.4\ \mu\text{r}\,. \end{aligned}$$

10.5. Consider a Gaussian wave with $R = 1$ m and $w = 10$ mm at a wavelength of 1 μm. Find R and w at a plane located 5 m away in the direction of propagation (i.e., the $+z$ direction).

We are given a Gaussian wave with $R_1 = 1$ m and $w_1 = 10$ mm at $\lambda = 1 \times 10^{-6}$ m and want to find the spot size w_2 and radius of curvature R_2 at a location 5 m to the right.

Assuming that the observation position is in the far-field, then the beam waist is located at

$$z_0' \approx R_1 = 1 \text{ m}. \tag{14a}$$

and

$$w_0 = \frac{\lambda z_0'}{\pi w_1} = \frac{(1 \times 10^{-6})(1)}{\pi (10 \times 10^{-3})} = 3.183 \times 10^{-5} \text{ m}. \tag{14b}$$

The Rayleigh range is

$$z_R = \frac{\pi w_0^2}{\lambda} = \frac{\pi (3.183 \times 10^{-5})^2}{1 \times 10^{-6}} = 3.183 \times 10^{-3} = 3.183 \text{ mm}. \tag{14c}$$

Since 1 m >> 3.183 mm, our far-field assumption was justified and we can now continue.

The beam waist is 1 m *to the left* of the current beam location. The location for the new data will be 6 m from the beam waist location. We note that we are in the far-field and that

$$R(z) = z = 6 \text{ m}, \tag{15a}$$

and

$$w(z) = \frac{\lambda z}{\pi w_0} = \frac{(1 \times 10^{-6})(6)}{\pi (3.183 \times 10^{-5})} = 6.00 \times 10^{-2} \text{ m} = 6.00 \text{ cm}. \tag{15b}$$

Alternative solution: It is also possible to solve this problem using the complex radius of curvature. We know that $\tilde{q}_2 = \tilde{q}_1 + 5$. Starting with $\tilde{q}_1$, we find

$$\begin{aligned} \frac{1}{\tilde{q}_1} &= \frac{1}{R} - j\frac{\lambda}{\pi w_1^2} = \frac{1}{1} - j\frac{1 \times 10^{-6}}{\pi (10 \times 10^{-3})^2} \\ &= 1 - j(3.18 \times 10^{-3}) = 1.00 \,\underline{/-0.182}\, . \end{aligned} \tag{16a}$$

$$\tilde{q}_1 = 1.00 \,\underline{/+0.182}\, = 0.999 + j(3.18 \times 10^{-3}) \tag{16b}$$

$$\begin{aligned} \tilde{q}_2 &= \tilde{q}_1 + 5 = 0.999 + j(3.18 \times 10^{-3}) + 5 \\ &= 5.999 + j(3.18 \times 10^{-3}) = 5.99 \,\underline{/+3.03 \times 10^{-3}} \end{aligned} \tag{17a}$$

$$\begin{aligned} \frac{1}{\tilde{q}_2} &= 5.99 \,\underline{/-3.03 \times 10^{-3}}\, = 0.1666 - j(8.82 \times 10^{-5}) \\ &= \frac{1}{R_2} - j\frac{\lambda}{\pi w_2^2} \end{aligned} \tag{17b}$$

$$\frac{1}{R_2} = 0.1666 \tag{18a}$$

$$R_2 = 6.02 \text{ m} \tag{18b}$$

$$-\frac{\lambda}{\pi w_2^2} = -8.82 \times 10^{-5} \tag{19a}$$

$$w_2 = \sqrt{\frac{(1.0 \times 10^{-6}}{\pi(8.82 \times 10^{-5})}} = 6.00 \times 10^{-2} \text{ m} = 6.00 \text{ cm}\,. \tag{19b}$$

10.7. For $D = 3w$, calculate the fraction of power transmitted through an aperture.

For $D = 3w$ (or $a = 1.5w$), we want to calculate the fraction of the power transmitted.

$$\begin{aligned} \frac{P(a)}{P_{\text{inc}}} &= 1 - \exp\left[-\frac{2a^2}{w^2}\right] = 1 - \exp\left[-\frac{2(1.5w)^2}{w^2}\right] \\ &= 1 - \exp(-2(1.5)^2) = 0.989 = 98.9\%\,. \end{aligned} \tag{20}$$